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On wave set-up in shoaling water with a rough sea bed

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In very gradually shoaling coastal water the energy of incident waves appears to be absorbed not by breaking at the upper surface, but predominantly by turbulent dissipation near the rippled sea bed. The question has been asked whether there can be then any wave set-up, that is any increase in the mean water level, at, or close to, the shoreline.

To answer this question the equations of wave energy and momentum in water of slowly varying depth are generalized so as to include the presence of a dissipative boundary layer at the bottom. It is then shown that the resulting equation for the mean surface slope can be integrated exactly, to give the mean surface depression (the 'set-down') in terms of the local wave amplitude and water depth, outside the surf zone. In the special case of a uniform beach slope s, a closed expression is obtained for the wave amplitude in terms of the local depth, under two different sets of conditions: (i) when the thickness of the boundary layer at the bottom is assumed to be constant, and (ii) for waves over a rippled bed, when the boundary-layer thickness corresponds to the measured dissipation of energy in oscillatory waves over steep sand ripples. In both cases it is found that there exists a maximum bottom slope s below which the wave amplitude must diminish monotonically towards the shoreline. This maximum value of s is of order 10^{-3}. The waves can indeed penetrate close to the shoreline without breaking, and the corresponding wave set-up is negligible. An example of where suitable conditions exist is on the continental shelf off North Carolina.

1. Introduction

On some shorelines such as the Atlantic coastline of the USA, the water shelves very gradually (see Herbers, Hendrickson & O'Reilly 2000), and most of the energy of incoming swell appears to be dissipated near to the rippled sea bed, not by wave breaking at the surface. A question put to some of his colleagues, including myself, by Professor R. Guza was this: In such a situation, would one expect to see the usual wave 'set-up' which occurs near a shoreline inside the breaker zone? It will be recalled that outside the breaker zone the wave set-up is actually negative: there is in fact a small wave 'set-down' (Saville 1961; Longuet-Higgins & Stewart 1963, 1964). If the wave energy is all dissipated near the bottom, and no breaking occurs, what happens to the wave set-up?

A further question is whether it is actually possible for waves to reach the shoreline without breaking at all, and if so, under what conditions.

Seawards of the breaker line an important part is normally played by the well-known phenomenon of mass transport in the bottom boundary layer (the

'Stokes layer'). As shown by Longuet-Higgins (1953), if the horizontal component of the oscillatory velocity of the fluid just beyond the boundary layer is given by

$$u = q\cos(kx - \sigma t) \tag{1.1}$$

where x is a horizontal (or tangential) coordinate, t is the time and k and σ are the wavenumber and radian frequency, then just outside the boundary layer there must be a horizontal Lagrangian velocity

$$U = \frac{5}{4}\frac{q^2}{c} \tag{1.2}$$

in the direction of propagation, where $c = \sigma/k$, the phase speed. This is of course quadratic in the orbital velocity q. Remarkably, although the result (1.2) was obtained by integration of the full Navier–Stokes equations of motion throughout the boundary layer, it is independent of the kinematic viscosity ν.

Furthermore, it has been shown that even if the viscosity, or the eddy viscosity, varies throughout the layer, equation (1.2) remains valid, to second order in the wave amplitude, provided the viscosity is a function only of the mean distance of a particle from the boundary (Longuet-Higgins 1957).

The 'bottom-drift' velocity (1.2) is found to be in agreement with laboratory measurements by Bagnold (1947) and by Russell & Osorio (1957) and is also confirmed by field observations. This suggests that under normal conditions where waves approach a shoreline in shallowing water, there is a shorewards drift of water (and perhaps sand) near the bottom outside the breaker zone. Inside the breaker zone, however, the bottom drift is reversed, as can be shown by simple experiments (Longuet-Higgins 1983). Thus at the breaker line the two oppposing bottom currents meet and a sand bar (the 'breaker-bar') tends to be formed.

Now waves over a rippled sea bed tend to generate vortices (for a theoretical description see Longuet-Higgins 1981) which spread upwards and can carry sediment or sand in suspension (the suspended load). It is reasonable to represent this fluid layer as turbulent, with a certain eddy viscosity and a density somewhat greater than that of clear sea water. Thus we may consider waves in a two-layer system, of a total depth which decreases steadily towards the shoreline. What happens when waves pass through such a system?

In some preliminary experiments with a two-layer system the upper layer was pure water and the lower layer was a sugar solution consisting of equal weights of water and brown sugar. The density of the mixture was measured to be 1.30 times that of pure water, and the viscosity, at room temperature, was determined as about 16 times that of water. The contrast in colour was sufficient, over the duration of the experiment, to define the boundary between the two fluids, although as time progressed some diffusion and mixing occurred. A description of the apparatus, the experimental procedure and the results have been given elsewhere (Longuet-Higgins 2004). Briefly, it was found that a bottom boundary-layer consisting mainly of fluid from the lower, denser layer, advances up the sloping beach as far as the breaker-line. There it rises vertically and is diffused by the turbulence from the breaker zone. If one reduces the amplitude of the incident waves, the breaker-line moves closer to the beach, and with it the bottom boundary-layer.

The present paper is essentially a theoretical discussion of whether it is possible for purely progressive waves, in such a situation, to reach a shoreline without breaking.

In §2 and §3 we generalize the discussion of energy and momentum flux in progressive, irrotational surface waves in water of slowly varying depth to the situation when there is a thin, oscillating boundary-layer (a Stokes layer) at the solid bottom. Expressions for the mean rate of energy dissipation D and for the mean horizontal stress τ_B on the bottom are derived and it is noted that

$$D = c\tau_B \tag{1.3}$$

where c is the phase speed. By using equation (1.3) to combine the equations of energy and momentum it is shown in § 4 that there exists an exact integral, equation (4.14), for the second-order change in mean level (the wave 'set-up') in terms of the local depth h and the local wave amplitude a, even in the presence of energy dissipation at the bottom. In § 5 this result is further generalized to the situation when the bottom boundary layer has a different fluid density ρ' and kinematic viscosity ν' than in the rest of the wave.

In § 6 we consider the special case when the bottom slope s is uniform. It is found that in shallow water, and provided the thickness δ' of the boundary layer is constant, there exists an analytic solution for the wave amplitude a as a function of the depth h or as a function of distance x from the shoreline. This is shown in figure 1.

The more common situation when the bottom consists of sand ripples is treated in § 7. By comparing the measured rates of energy dissipation for waves over a rippled bed with the energy dissipation in a Stokes layer one can derive a simple expression, equation (7.1), for the equivalent thickness of the bottom boundary layer. This also is supported theoretically by vortex-shedding calculations (see Appendix B). Owing to variations in ripple steepness and sand-grain diameter the constant P in equation (7.1) is determined only to an order of magnitude. Interestingly, however, it turns out that the resulting equation for the wave amplitude a as a function of the depth can be solved exactly in analytical form. The result is plotted graphically in figures 2(a) and 2(b).

One consequence of the solutions in both § 6 and § 7 is that there exists a maximum bottom slope s for which the ratio a/h always decreases steadily towards the shoreline, so that if the waves are not breaking at a certain depth h_0 then they will not break when $h < h_0$. This critical bottom slope is of order 10^{-3}; see equation (7.18).

In § 8 we consider as an example the wave observations of Herbers *et al.* (2000) over the continental shelf off North Carolina, and show that that situation is consistent with the theory of § 7. A discussion and conclusions follow in § 9 and § 10.

2. Energy flux

In the following analysis, which applies to waves outside the surf zone when the bottom slope is small and there is no reflection from the shoreline, we shall make use of the linearized theory of surface waves in water of uniform depth. In previous studies, e.g. Longuet-Higgins & Stewart (1964), this has provided surprisingly accurate results. In the present instance the theory has to be slightly modified by the presence of a boundary layer at the bottom, in which the tangential velocity is given by

$$u = q\mathrm{e}^{\alpha(z+h)+\mathrm{i}(kx-\sigma t)} \tag{2.1}$$

where q is the amplitude of the velocity just outside the oscillatory Stokes boundary layer; see § 1. q is related to the wave amplitude a at the surface ($2a$ is the wave height) by

$$q = \frac{a\sigma}{\sinh kh} \tag{2.2}$$

and α is given by

$$\nu\alpha^2 = -\mathrm{i}\sigma, \quad \mathrm{Re}(\alpha) > 0 \tag{2.3}$$

so

$$\alpha = (1-\mathrm{i})/\delta \quad \text{where} \quad \delta = (2\nu/\sigma)^{1/2}. \tag{2.4}$$

Here ν is the kinematic eddy viscosity which for simplicity will be assumed to be a constant. The nominal thickness δ of the boundary layer will be assumed to be small compared to the wavelength $2\pi/k$ and also the mean depth h, so that both

$$k\delta \ll 1 \quad \text{and} \quad \delta/h \ll 1. \tag{2.5}$$

The dispersion relation

$$\sigma^2 = gk \tanh kh \tag{2.6}$$

for gravity waves on an inviscid fluid is then closely valid.

The density of kinetic energy in the boundary layer, per unit horizontal distance, is given by

$$E_B = \int_{-h}^{-h+0(\delta)} \tfrac{1}{2}\rho u^2 \mathrm{d}z = \tfrac{1}{8}\rho q^2 \delta \tag{2.7}$$

with neglect of $(k\delta)^2$. In comparison with the inviscid wave energy density

$$E = \tfrac{1}{2}\rho g a^2 \tag{2.8}$$

E_B is clearly small, by the assumptions (2.5). Thus the horizontal flux of energy F can be written, to this approximation, as

$$F = Ec_g \tag{2.9}$$

where E is given by (2.8) and c_g denotes the group velocity:

$$c_g = \frac{\sigma}{2k}\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{2.10}$$

In a quasi-steady state, the energy balance equation becomes

$$\frac{\mathrm{d}F}{\mathrm{d}x} = -D \tag{2.11}$$

where D denotes the rate of energy dissipation per unit area within the boundary layer, that is

$$D = \int_{-h}^{-h+0(\delta)} \rho\nu \left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}z = \tfrac{1}{4}\rho\sigma q^2 \delta. \tag{2.12}$$

In view of equation (2.2) and the dispersion relation (2.6) this can be written

$$D = \tfrac{1}{4}\rho g a^2 \sigma \frac{2kh}{\sinh 2kh}(\delta/h). \tag{2.13}$$

On substituting the expressions (2.8), (2.9) and (2.10) into equation (2.11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho g a^2 \frac{\sigma}{4k}\left(1 + \frac{2kh}{\sinh 2kh}\right)\right] = -D \tag{2.14}$$

where D is given by equation (2.13).

3. Momentum flux

We shall now consider the change in the mean surface level $\bar{\zeta}$ accompanying the variation in wave energy described in § 2. This can be derived from the horizontal flux of horizontal momentum, by a generalization of the argument given in Longuet-Higgins & Stewart (1962, 1964).

In a quasi-steady state the balance of horizontal momentum is given, to lowest order, by

$$\frac{\mathrm{d}S_{xx}}{\mathrm{d}x} + \rho g h \frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}x} + \tau_B = 0 \tag{3.1}$$

where S_{xx} denotes the radiation stress component

$$S_{xx} = \tfrac{1}{2}\rho g a^2 \left(\frac{1}{2} + \frac{2kh}{\sinh 2kh} \right) \tag{3.2}$$

and τ_B is the mean tangential stress exerted by the waves on the bottom. Since the mass-transport velocity U_B in the boundary layer (see Longuet-Higgins 1953) is

$$U_B = \frac{q^2}{4c}\left(5 - 8\mathrm{e}^{-(z+h)/\delta}\cos(z+h)/\delta + 3\mathrm{e}^{-2(z+h)/\delta}\right) \tag{3.3}$$

where $c = \sigma/k$, we find

$$\left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)_{z=-h} = \frac{q^2}{2c\delta} \tag{3.4}$$

Thus

$$\tau_B = \rho\nu \left(\frac{\mathrm{d}U_B}{\mathrm{d}z}\right)_{z=-h} = \frac{\rho\nu k q^2}{2\sigma\delta} = \tfrac{1}{4}\rho k q^2 \delta \tag{3.5}$$

since δ is given by (2.4). From (2.12) we notice that

$$D = c\tau_B \tag{3.6}$$

as might be expected. Thus equation (3.1) can be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tfrac{1}{2}\rho g a^2 \left(\frac{1}{2} + \frac{2kh}{\sinh 2kh}\right)\right] + \rho g h \frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}x} = -D/c. \tag{3.7}$$

The relation (3.6) between the mean dissipation D and the mean bottom stress τ_B is not surprising when we consider the fluid motion relative to axes moving horizontally with the phase speed c. For then the bottom boundary layer appears as the result of a horizontal stress τ_B moving with speed $-c$ and applied to an otherwise stationary fluid. The energy and momentum transferred to the fluid should then be in the ratio c.

This argument leads us to expect that equation (3.6) will apply even when the viscosity (or the eddy viscosity) in the boundary layer is not a constant, so that the profile of the mass transport is not necessarily given by equation (3.3).

We can combine the momentum flux equation with the energy flux equation (2.14) by multiplying each side of (3.7) by c and subtracting the result from (2.14). Thus we obtain

$$h\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}x} = \tfrac{1}{4}k\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{a^2}{k}\left(1 + \frac{2kh}{\sinh 2kh}\right)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\tfrac{1}{2}a^2\left(\frac{1}{2} + \frac{2kh}{\sinh 2kh}\right)\right] \tag{3.8}$$

where a^2, k and h are all considered to be slowly varying functions of x. Note that since $\mathrm{d}h/\mathrm{d}x = -s$, where s is the local bottom slope, not necessarily constant, we can, after dividing each side of (3.8) by s, replace $\mathrm{d}/\mathrm{d}x$ by $\mathrm{d}/\mathrm{d}h$ throughout, or simply by the differential d.

We shall now show that equation (3.8) has an exact integral for $\bar{\zeta}$ in terms of a^2, k and h.

4. An exact integral of equation (3.8)

In this proof we follow the method of Longuet-Higgins & Stewart (1962, §4), but generalize to when the energy flux F is no longer a constant.

From (2.9), (2.10) and (3.2), equation (3.8) can be written

$$\mathrm{d}\bar{\zeta} = -\frac{1}{\rho g h}(dS_{xx} - c^{-1}\mathrm{d}F) \tag{4.1}$$

where

$$S_{xx} = F\left(\frac{2}{c} - \frac{1}{2c_g}\right). \tag{4.2}$$

From the general formula

$$c_g = \frac{\partial \sigma}{\partial k} = \frac{1}{2\sigma}\frac{\partial \sigma^2}{\partial k} \tag{4.3}$$

(where h is kept constant during partial differentiation) we have then

$$S_{xx} = \sigma F\left(\frac{2k}{\sigma^2} - \frac{\partial k}{\partial \sigma^2}\right). \tag{4.4}$$

Now in terms of the dimensionless variables

$$\xi = kh, \quad \eta = \sigma^2 h/g \tag{4.5}$$

the dispersion relation (2.6) becomes simply

$$\eta = \xi \tanh \xi \tag{4.6}$$

and so

$$\left.\begin{aligned} \frac{\sigma^2}{k} &= g \tanh kh = g\frac{\eta}{\xi}, \\ \frac{\partial \sigma^2}{\partial k} &= \frac{\partial(g\eta/h)}{\partial(\xi/h)} = g\frac{\mathrm{d}\eta}{\mathrm{d}\xi}. \end{aligned}\right\} \tag{4.7}$$

Thus equation (4.4) can be written

$$S_{xx} = \frac{\sigma F}{g}\left(\frac{2\xi}{\eta} - \frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right). \tag{4.8}$$

Substituting in equation (4.1) we have

$$\mathrm{d}\bar{\zeta} = -\frac{\sigma^3}{\rho g^3}\frac{1}{\eta}\left[\mathrm{d}\left\{F\left(\frac{2\xi}{\eta} - \frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)\right\} - \frac{\xi}{\eta}\mathrm{d}F\right]. \tag{4.9}$$

After simplification we obtain

$$\mathrm{d}\bar{\zeta} = \frac{\sigma^3}{\rho g^3}\frac{1}{\eta}\left[F\mathrm{d}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta} - \frac{2\xi}{\eta}\right) + \mathrm{d}F\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta} - \frac{\xi}{\eta}\right)\right] \tag{4.10}$$

of which the integral is

$$\bar{\zeta} = \frac{\sigma^3}{\rho g^3}F\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\xi}{\eta}\right) \tag{4.11}$$

see Appendix A. But from equations (2.9) and (4.3)

$$F = \frac{E}{2\sigma}\frac{\partial\sigma^2}{\partial k} = \frac{Eg}{2\sigma}\frac{d\eta}{d\xi}. \tag{4.12}$$

Therefore

$$\bar{\zeta} = \frac{\sigma^2 E}{2\rho g^2}\frac{d}{d\xi}\coth\xi \tag{4.13}$$

and on using (2.6) we obtain

$$\bar{\zeta} = -\tfrac{1}{2}\frac{a^2 k}{\sinh 2kh}, \tag{4.14}$$

the desired result. Since the right-hand side of (4.14) is always negative, we have in general a wave set-down.

In shallow water ($kh \ll 1$) equation (4.14) becomes

$$\bar{\zeta} = -\frac{a^2}{4h}. \tag{4.15}$$

5. Waves in a two-fluid system

We shall now extend the previous results to include the more general situation when the fluid in the bottom boundary layer is of a different density ρ' and kinematic viscosity ν' to that in the rest of the wave.

Since the Stokes layer at the bottom is driven essentially by the horizontal pressure gradient on its upper side, the fluid velocities within the layer are similar to that in a boundary layer of density ρ, except that they are reduced in magnitude by a factor ρ/ρ'. The vertical scale of the motion will be changed by a factor $(\nu'/\nu)^{1/2}$. By continuity, there will be an extra vertical velocity just outside the boundary layer of order $(\rho/\rho' - 1)qk\delta'$ which will modify the wave motion above the layer by an amount of order $qk\delta'$ only. To lowest order in $k\delta$, q in equation (2.7) is replaced by $(\rho/\rho')q$ and δ is replaced by $\delta' = (\nu'/\nu)^{1/2}\delta$. Instead of equation (2.12) we now have

$$D = \tfrac{1}{4}\rho'\sigma(\rho/\rho')^2 q^2\delta' \tag{5.1}$$

to lowest order. In the energy balance equation there will be an additional term of order $(\rho' - \rho)g(dh/dx)U\delta'$ arising from the energy required to propel the denser fluid up the slope, but since we assume dh/dx to be very small this term can be neglected.

Similarly, in equation (3.3) the expression for the mass-transport velocity in the boundary layer becomes

$$U_B' = \frac{(\rho/\rho')^2 q^2}{4c}\left(5 - 8e^{-(z+h)/\delta'}\cos(z+h)/\delta' + 3e^{-2(z+h)/\delta'}\right). \tag{5.2}$$

In equation (3.4) the vertical gradient of the mass-transport velocity when $z = 0$ is now $(\rho/\rho')^2 q^2/2c\delta'$ and hence in equation (3.5) we find for the mean bottom stress

$$\tau_B = \frac{\rho'\nu' k(\rho/\rho')^2 q^2}{2\sigma\delta'} = \tfrac{1}{4}\rho' k(\rho/\rho')^2 q^2\delta'. \tag{5.3}$$

Comparing equations (5.1) and (5.3) we see that $D = c\tau_B$ as before. Hence we can still combine the energy and momentum equations and the integral (4.14) is still valid.

Note that equation (4.15) seems to answer one of our original questions. For, if the wave amplitude a tends to zero at the shoreline (or, more precisely if $a\sigma^2/g \to 0$)

and if the wave does not break, implying that a/h is always bounded, then it follows from equation (4.15) that

$$\bar{\zeta} = -\tfrac{1}{4}a(a/h) \to 0 \tag{5.4}$$

at the shoreline. In other words, the wave set-up does indeed tend to zero. However, we have not yet shown the hypothesis $a(\sigma^2/g) \to 0$ to be consistent with all of our other assumptions, in particular that $\delta'/h \ll 1$.

6. The wave amplitude

To determine the variation of the wave amplitude a in a particular instance we must return to equation (2.14) which, when D is given by (5.1), may be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{a^2}{k}\left(1+\frac{2kh}{\sinh 2kh}\right)\right] = -(\rho/\rho')a^2\frac{2kh}{\sinh 2kh}(\delta'/h). \tag{6.1}$$

In shallow water, that is when $kh \ll 1$, this reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2a^2}{k}\right) = -\frac{(\rho/\rho')a^2\delta'}{h} \tag{6.2}$$

and since in shallow water $k = \sigma/(gh)^{1/2}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^2h^{1/2}\right) = -\frac{(\rho/\rho')a^2\sigma\delta'}{2g^{1/2}h}. \tag{6.3}$$

In the inviscid case when $\delta' = 0$ the right-hand side vanishes and we recover Green's law

$$a \propto h^{-1/4} \tag{6.4}$$

for waves entering shallow water.

In the general case when $\delta' \neq 0$ let us assume for simplicity that the bottom slope s is constant. Then we have

$$h = -sx, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = -s\frac{\mathrm{d}}{\mathrm{d}h} \tag{6.5}$$

and equation (6.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(a^2h^{1/2}\right) = \frac{1}{2}C\frac{a^2}{h} \tag{6.6}$$

where

$$C = \frac{\rho}{\rho'}\frac{\sigma\delta'}{g^{1/2}s}. \tag{6.7}$$

If C is assumed to be a constant, we find, by substituting $a^2h^{1/2} = y$, that equation (6.6) can be integrated to give

$$a^2 = \frac{B}{h^{1/2}}\mathrm{e}^{-C/h^{1/2}} \tag{6.8}$$

where B is a constant, to be determined by the initial conditions at some depth $h = h_0$. Note that as $h \to \infty$, so the right-hand side of (6.8) tends to zero.

To avoid breaking, we are interested in the maximum value of the parameter a/h. Writing $\lambda = a/h$ we have from equation (6.8)

$$(\lambda/\lambda_0)^2 = (h_0/h)^{5/2}\exp\left(-C\left(h^{-1/2} - h_0^{-1/2}\right)\right). \tag{6.9}$$

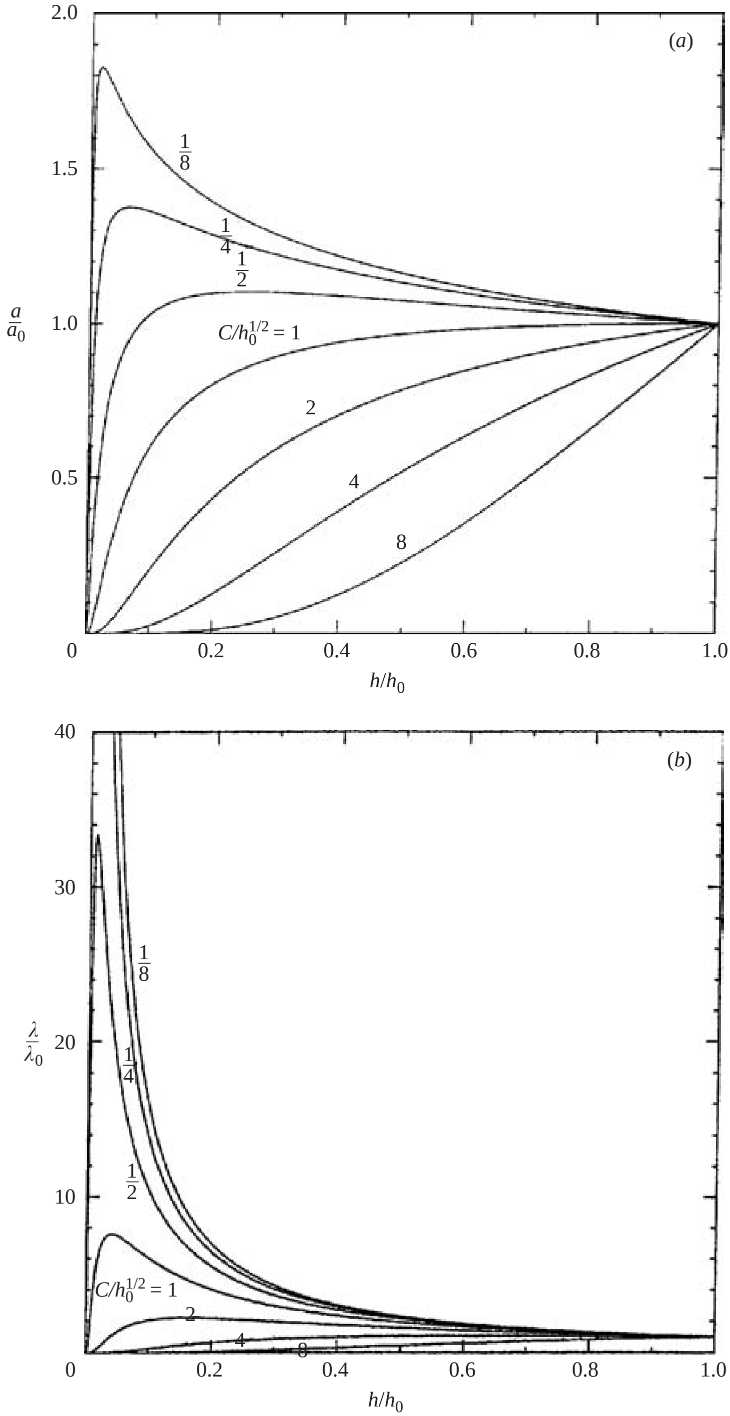

FIGURE 1. Plots of (*a*) the relative wave amplitude and (*b*) the wave steepness as given by the solution (6.8).

But the maximum value of $h^{-5/2}\exp(-Ch^{-1/2})$ is $(5/C\mathrm{e})^5$, occurring when $Ch^{-1/2}=5$. So the maximum value $\lambda_{\max}$ is given by

$$(\lambda_{\max}/\lambda_0)^2 = \left(5h_0^{1/2}/C\mathrm{e}\right)^5 \exp\left(C/h_0^{1/2}\right). \tag{6.10}$$

Graphs of a/a_0 and of λ/λ_0 corresponding to equations (6.8) and (6.9) are shown in figures 1(*a*) and 1(*b*) respectively, for given values of the parameter $C/h_0^{1/2}$ and $C_0/h_0^{1/2}$. It will be seen that for values of $C/h_0^{1/2}$ greater than 1 the wave height diminishes monotonically as the depth h diminishes, and for values of $C/h_0^{1/2}$ greater than 5 the wave steepness λ also decreases monotonically.

7. Waves over a rippled sea bed

The question now is, what value should we assume for the boundary-layer thickness δ'? From a practical viewpoint the most interesting situation is when the bottom is covered with sand ripples. From both laboratory measurements of the damping of an oscillatory flow over steep sand ripples (Carstens, Nielson & Altinbilek 1969) and from numerical calculations using a vortex-shedding technique (Longuet-Higgins 1981) it appears that the mean drag coefficient $\bar{C}_D$ over steep sand ripples is generally of order 0.1 (see Appendix B). The consequent rate of dissipation is comparable to that in an oscillatory Stokes layer, provided that δ' is of order $0.16b$, where $2b$ is the horizontal excursion of a fluid particle associated with the wave motion, that is $b=a/\sinh kh$; see equations (B6) and (B7). Hence in equation (6.7), instead of taking δ' as constant, it may be more appropriate to assume that

$$\delta' = 2P\,\frac{a}{\sinh kh}, \tag{7.1}$$

where a is now the *local* wave amplitude and P is a constant of order 0.08. Then in shallow water we have

$$\delta' = 2P\,\frac{a}{kh} = 2P\,\frac{a(g/h)^{1/2}}{\sigma} \tag{7.2}$$

and equation (6.6) becomes simply

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(a^2h^{1/2}\right) = P\frac{\rho/\rho'}{s}\frac{a^3}{h^{3/2}}. \tag{7.3}$$

Substituting $a^2h^{1/2}=y$ as before we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}h} = Q\frac{y^{3/2}}{h^{9/4}}, \qquad Q = \frac{P(\rho/\rho')}{s} \tag{7.4}$$

or

$$y^{-3/2}\mathrm{d}y = Qh^{-9/4}\mathrm{d}h. \tag{7.5}$$

Hence

$$y^{-1/2} = \tfrac{2}{5}Qh^{-5/4} + B_1 \tag{7.6}$$

where B_1 is an arbitrary constant. The last equation can be written

$$\frac{1}{a} = \tfrac{2}{5}Qh^{-1} + B_2h^{1/4} \tag{7.7}$$

or more conveniently

$$a = \tfrac{5}{2}Q^{-1}h/\left(1 + B_3(h/h_0)^{5/4}\right). \tag{7.8}$$

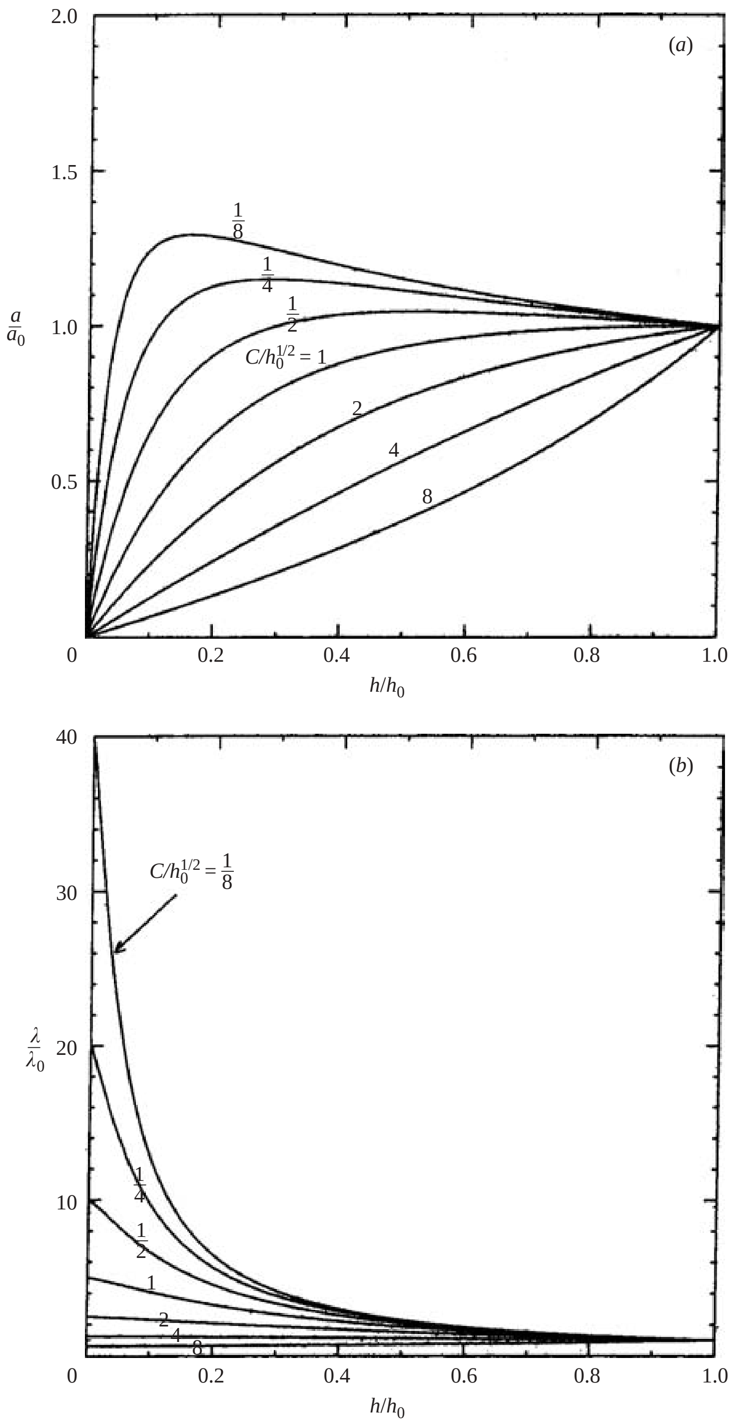

FIGURE 2. Plots of (*a*) the relative wave amplitude and (*b*) the wave steepness as given by the solution (7.9).

The arbitrary constant B_3 is to be chosen so as to satisfy the initial condition that $a = a_0$ when $h = h_0$. Equation (7.7) can also be written as

$$\frac{a}{a_0} = \frac{5}{C_0/h_0^{1/2}} \frac{h/h_0}{1 + B(h/h_0)^{5/4}} \tag{7.9}$$

so that on writing $a/a_0 = h/h_0 = 1$ we find

$$B = \frac{5}{C_0/h_0^{1/2}} - 1. \tag{7.10}$$

The solution thus depends essentially on the value of the parameter $C_0/h_0^{1/2}$. From equations (6.7) and (7.2) this is given by

$$C_0/h_0^{1/2} = \frac{2P}{s} \frac{\rho}{\rho'} \frac{a_0}{h_0}. \tag{7.11}$$

Using equation (7.9), the functions a/a_0 and $\lambda/\lambda_0 = (a/h)/(a_0/h_0)$ are plotted against h/h_0 in figure 2 for given values of $C_0/h_0^{1/2}$. Comparing the curves with those in figure 1 we see that qualitatively they are very similar.

In particular, the wave amplitude a, given by equation (7.9), is a maximum when

$$(h/h_0)^{5/4} = \frac{4}{5/\left(C_0/h_0^{1/2}\right) - 1} \tag{7.12}$$

provided $C_0/h_0^{1/2} < 5$, but this maximum lies in the range $0 < h < h_0$ only when $C_0/h_0^{1/2} < 1$. If $C_0/h_0^{1/2} > 1$, then a/a_0 decreases monotonically towards zero as the depth diminishes. In the limit as $h/h_0 \to 0$ we have

$$\frac{a}{a_0} \sim \frac{5}{C_0/h_0^{1/2}}(h/h_0). \tag{7.13}$$

On the other hand the relative wave steepness, given by

$$\frac{\lambda}{\lambda_0} = \frac{a/h}{a_0/h_0} = \frac{5}{C_0/h_0^{1/2}} \frac{1}{1 + B(h/h_0)^{5/4}}, \tag{7.14}$$

always decreases monotonically as the depth diminishes provided that $B < 0$, that is

$$C_0/h_0^{1/2} > 5. \tag{7.15}$$

When $C_0/h_0^{1/2} = 5$ then $B = 0$ and $\lambda/\lambda_0 = 1$, a constant. The criterion (7.15) is quite similar to that found for the model of § 6 (in which δ' was assumed uniform) despite the different analytic form of the solution.

The inequality (7.15) suggests that there will certainly be no wave breaking at the upper surface of the sea if $C_0/h_0^{1/2} > 5$, or from equation (7.11)

$$\frac{2P}{s} \frac{\rho}{\rho'} \frac{a_0}{h_0} > 5 \tag{7.16}$$

hence

$$s < \frac{2P}{5} \frac{\rho}{\rho'} \frac{a_0}{h_0}. \tag{7.17}$$

Since $2P/5 = 0.032$, $\rho/\rho' < 1$ and $4a_0$ (i.e. twice the significant wave height) is less than 0.83 h_0 (the limiting height of a solitary wave) we have

$$s < 0.032 \times 0.21 = 6 \times 10^{-3} \tag{7.18}$$

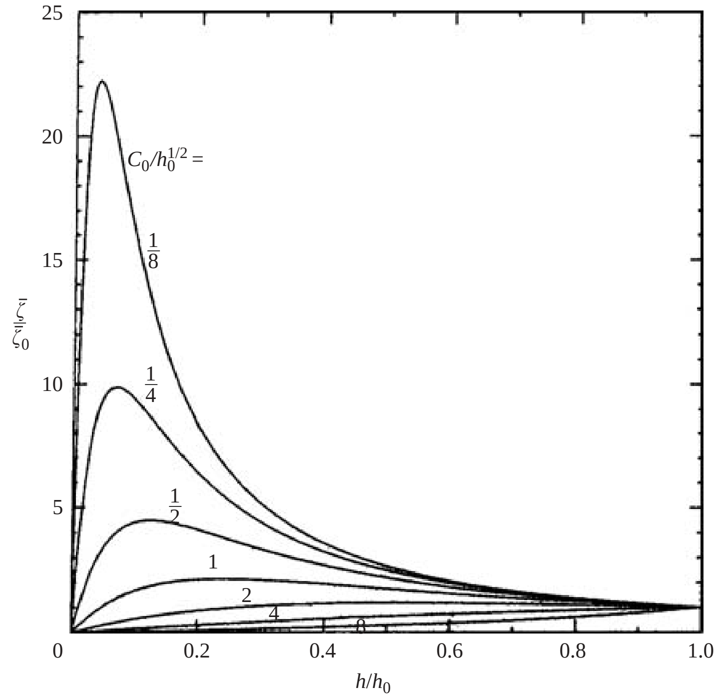

FIGURE 3. Plot of the relative set-down, as given by equation (7.20).

as a maximum value of the bottom slope if breaking at the surface is to be always avoided.

The corresponding wave set-down is given by equation (4.15), so that the relative wave set-down, namely

$$\bar{\zeta}/\bar{\zeta}_0 = \frac{(a/a_0)^2}{h/h_0} = (a/a_0)(\lambda/\lambda_0), \tag{7.19}$$

is found as the product of (7.12) and (7.13), namely

$$\bar{\zeta}/\bar{\zeta}_0 = \left(\frac{5}{C_0/h_0^{1/2}}\right)^2 \frac{h/h_0}{\left[1 + B(h/h_0)^{5/4}\right]^2}. \tag{7.20}$$

This has a maximum value in the range $0 < h < h_0$ provided that $C_0/h_0^{1/2} < 3$; see figure 3. But if $C_0/h_0^{1/2} > 3$ then $\bar{\zeta}/\bar{\zeta}_0$ always diminishes towards shallower water.

The ratio $\bar{\zeta}/h$ is also of some interest since for the validity of the assumptions this should be small. From equation (7.1) we find

$$\frac{\bar{\zeta}/h}{\bar{\zeta}_0/h_0} = \frac{(a/a_0)^2}{(h/h_0)^2} = \left(\frac{a/h}{a_0/h}\right)^2. \tag{7.21}$$

This is plotted in figure 4. It is in fact the square of the quantity plotted in figure 2(*b*). When $C_0/h_0^{1/2} > 5$ it always diminishes towards the shoreline.

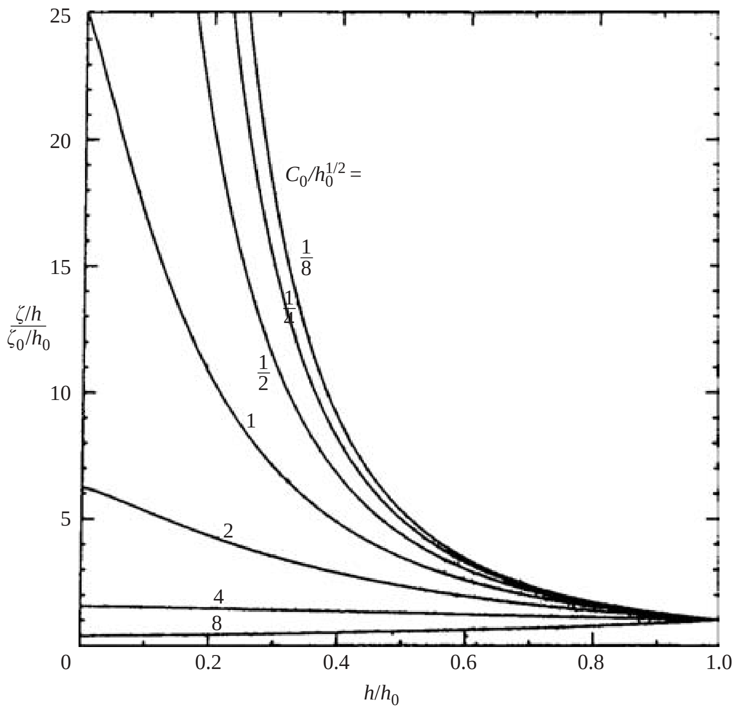

FIGURE 4. The wave set-down $\bar{\zeta}$ as a proportion of the total depth h, relative to the value at $h = h_0$.

8. A practical example

As an example consider the observations made by Herbers *et al.* (2000) of swell approaching the coast of North Carolina. A brief indication of the depth profile across the shallow continental shelf (as accurately as can be read from figure 1 of Herbers *et al.* 2000) is given in table 1. Here x is the distance of a station from the shoreline and h is the local depth. The value of $\sigma^2 h/g$ is calculated from the representative values of the wave frequency $f = 2\pi/\sigma$ and then kh is found from the dispersion relation (2.6). At stations A and B it can be seen that the shallow-water approximation $kh = \sigma(h/g)^{1/2}$ is roughly applicable.

Let us then apply the model to the waves between Station B ($h = 20$ m) and Station A ($h = 12$ m), assuming that the waves are two-dimensional and that their direction is normal to the coastline. From figure 6 of Herbers *et al.* (2000) the measured swell variance at Station B on October 19, 1994 (Julian day 292) was about 1000 cm^2, corresponding to a significant wave height $2a_0 = 4 \times (1000)^{1/2}$ cm $= 1.26$ m. Thus we have

$$h_0 = 20\,\text{m}, \qquad a_0 = 0.65\,\text{m}. \tag{8.1}$$

The peak frequency of the waves was 0.07 Hz (see their figure 7). The mean gradient s of the sea bed between Stations A and B was 1.3×10^{-3}. From equation (7.11), assuming $2P = 0.16$ and $\rho/\rho' = 0.8$, we have then

$$C_0/h_0^{1/2} = 3.2. \tag{8.2}$$

Station	x (km)	h (m)	$\sigma^2 h/g$		kh		$2a$ (m)	a/h
			$f=0.10$	0.07	0.10	0.07		
A	1.5	12	0.48	0.24	0.75	0.51	1.40	0.058
B	8	20	0.80	0.40	1.03	0.68	1.26	0.032
C	16	26	1.04	0.52	1.23	0.79	1.38	0.026
H	92	47	1.88	0.94	1.96	1.15	2.37	0.025

TABLE 1. Data from figures 1 and 2 of Herbers *et al.* (2000).

The relative increase in the theoretical wave steepness a/h would therefore be

$$5/\left(C_0/h_0^{1/2}\right) = 1.6 \tag{8.3}$$

compared to the observed increase, namely

$$\frac{0.058}{0.032} = 1.9. \tag{8.4}$$

Between Station A and the coastline ($x=0$) the mean bottom slope s is substantially greater: $s = 8 \times 10^{-3}$. Thus $C_0/h_0^{1/2}$ rises to 0.9, and $5/(C_0/h_0^{1/2}) = 5.4$. Hence $2a/h$ would have been as high as 0.6, which would lead probably to wave breaking. No wave measurements shorewards of Station A are reported by Herbers *et al.* (2000), so no direct comparison with observation can be made in this case. Nevertheless from figure 2 of their paper it is clear that on several occasions during the 100-day period of observation the significant wave height was of order 0.15 times the values quoted in table 1. Then $2a/h$ would have been less than 0.1. In such situations the waves would be unlikely to break and the set-down, given by equation (5.4), would be negligibly small.

9. Discussion

One of the assumptions in the above theoretical model is that the thickness δ' of the boundary layer is everywhere small compared to the mean depth h. Let us examine whether this assumption is justified. From equation (7.2) we have in general

$$\frac{\delta'}{h} = 2P\frac{(g/h)^{1/2}}{\sigma}\frac{a}{h} \tag{9.1}$$

where $\sigma = 2\pi f$ is the radian frequency of the waves. From the data of table 1 we find, at Station A, that

$$\frac{\delta'_0}{h_0} = 0.017. \tag{9.2}$$

But from equation (9.1) we also have

$$\frac{\delta'/h}{\delta'_0/h_0} = \frac{1}{(h/h_0)^{1/2}}\frac{a/h}{a_0/h_0}. \tag{9.3}$$

Hence if $\delta'/h > \epsilon$, say, we must have

$$\frac{a/h}{a_0/h_0} > \frac{\epsilon}{0.017}(h/h_0)^{1/2}. \tag{9.4}$$

The critical depth at which $\delta'/h = \epsilon$ can be found as the intersection of the curve $Y = \lambda/\lambda_0$ in figure 2(b) (corresponding to $C_0/h_0^{1/2} = 0.90$) with the curve

$Y = (\epsilon/0.017)(h/h_0)^{1/2}$. If for example we take $\epsilon = 0.2$ we find $h/h_0 = 0.15$, and since $h_0 = 12$ m, then $h = 1.8$ m. In other words, in order that the ratio δ'/h shall exceed 0.2, the depth h must be less than 1.8 m.

Shorewards of this point the present theory cannot strictly be applied. Nevertheless, provided that the waves are low enough not to break, we may expect qualitatively similar results.

10. Conclusions

From § 6 to § 9 we conclude that it is indeed possible for wave energy to be almost completely dissipated by turbulence or friction in the bottom boundary layer before reaching the shoreline, and that this is favoured if the bottom slope s is of order 10^{-3} or less. In the field observations by Herbers *et al.* (2000) on the continental shelf of North Carolina, the wave energy reaching the shoreline would nearly always have been small, and the corresponding wave set-up would have been negligible. However during the event of October 19, 1994, the waves were probably breaking at the shoreline, producing significant set-up.

I am indebted to Professor R. Guza for posing the initial question which led to this investigation. The theoretical analysis in § 2 to § 6 was stimulated by a question from Dr M. J. Buckingham at a physical oceanography seminar at the Scripps Institution on November 19, 2003.

Appendix A. Proof of equation (4.11)

We shall prove equation (4.11) by showing that the derivative of the right-hand side of equation (4.11) is equal to the right-hand side of (4.10). Thus

$$\begin{aligned} \mathrm{d}\left[F\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\xi}{\eta}\right)\right] &= \mathrm{d}F\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\xi}{\eta}\right) + F\mathrm{d}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\xi}{\eta}\right)\right] \\ &= \mathrm{d}F\left(\frac{1}{\eta}\frac{\mathrm{d}\xi}{\mathrm{d}\eta} - \frac{\xi}{\eta^2}\right) + F\mathrm{d}\left(\frac{1}{\eta}\frac{\mathrm{d}\xi}{\mathrm{d}\eta} - \frac{\xi}{\eta^2}\right) \\ &= \frac{1}{\eta}\mathrm{d}F\left(\frac{\mathrm{d}\xi}{\eta} - \frac{\xi}{\eta}\right) + FR \end{aligned} \tag{A 1}$$

where

$$\begin{aligned} R &= -\frac{1}{\eta^2}\mathrm{d}\eta\frac{\mathrm{d}\xi}{\mathrm{d}\eta} + \frac{1}{\eta}\mathrm{d}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right) - \frac{1}{\eta^2}\mathrm{d}\xi + \frac{2\xi}{\eta^3}\mathrm{d}\eta \\ &= -\frac{2}{\eta^2}\mathrm{d}\xi + \frac{1}{\eta}\mathrm{d}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right) + \frac{2\xi}{\eta^3}\mathrm{d}\eta \\ &= \frac{1}{\eta}\mathrm{d}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta} - \frac{2\xi}{\eta}\right). \end{aligned} \tag{A 2}$$

Then equation (4.10) follows.

Appendix B. Energy dissipation by flow over sand ripples

To estimate the dissipation of energy by wave motion over a rippled sand bed we make use of the laboratory measurements by Carstens *et al.* (1969), supported by the theoretical calculations of Longuet-Higgins (1981).

	$\overline{C_D}$ obs.	$\overline{C_D}$ theor.
(*a*) coarse	0.13	0.12
(*b*) medium	0.07	0.09
(*c*) fine	0.05	0.07

TABLE 2. Median values of the drag coefficient $\overline{C_D}$.

The instantaneous drag coefficient C_D may be defined by

$$\tau/\rho' = C_D U|U| \tag{B 1}$$

where U denotes the horizontal velocity just outside the turbulent boundary layer. The corresponding energy dissipation D, averaged over a cycle, is given by

$$D/\rho' = \overline{C_D U^2|U|} \tag{B 2}$$

and in case C_D varies over a cycle we write this as $\overline{C_D}\ \overline{U^2|U|}$ so defining the mean drag coefficient $\overline{C_D}$ by

$$\overline{C_D} = D/\rho'\overline{|U|^3}. \tag{B 3}$$

When U varies sinusoidally

$$U = U_0 \sin(\sigma t + \phi) \tag{B 4}$$

to lowest order, then $\overline{U^3} = (4/3\pi)U_0^3$ and equation (B2) becomes

$$D/\rho' = \frac{4}{3\pi}\overline{C_D}U_0^3. \tag{B 5}$$

Table 2 shows typical (i.e. median) values of $\overline{C_D}$ obtained by Carstens *et al.* (1969) for three different grades of sand. The right-hand column gives corresponding theoretical values calculated by Longuet-Higgins (1981). As a representative value we may take $\overline{C_D} = 0.1$.

Compare this model to the Stokes layer of §2 in which the mean dissipation $\overline{D}$ was given by equation (2.12). Writing ρ' and δ' for ρ and δ and q for U_0 we find that equations (2.12) and (B2) are equivalent provided that

$$\delta' = 0.1 \times \frac{16}{3\pi}\frac{q}{\sigma} = 0.16b \tag{B 6}$$

where

$$b = \frac{q}{\sigma} = \frac{a}{\sinh kh} \tag{B 7}$$

is the amplitude of the horizontal displacement of the fluid just outside the boundary layer.

As a check, note that the horizontal excursion $2b$ is typically about double the length, or pitch L, of the ripple (see Longuet-Higgins 1981, table 2) and that numerical simulations show the vortices thrown off by the ripple rising to levels comparable to L. Since the thickness of the boundary layer is of order $5\delta'$, the two estimates of δ' are in rough agreement.

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